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The holomorphic tension of vortices

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Abstract: We study the tension of vortices in $\mathcal{N}=2$ SQCD broken to $\mathcal{N}=1$ by a superpotential $W(\Phi)$, in color-flavor locked vacua. The tension can be written as $T = T_{\rm BPS} + T_{\rm non\,BPS}$. The BPS tension is equal to $4\pi|\mathcal{T}|$ where we call $\mathcal{T}$ the *holomorphic tension*. This is directly related to the central charge of the supersymmetry algebra. Using the tools of the Cachazo-Douglas-Seiberg-Witten solution we compute the holomorphic tension as a holomorphic function of the couplings, the mass and the dynamical scale: $\mathcal{T} = \sqrt{W'^2+f}$. A first approximation is given using the generalized Konishi anomaly in the semiclassical limit. The full quantum corrections are computed in the strong coupling regime using the factorization equations that relate the $\mathcal{N}=2$ curve to the $\mathcal{N}=1$ curve. Finally we study the limit in which the non-BPS contribution can be neglected because small with respect to the BPS one. In the case of linear superpotential the non-BPS contribution vanishes exactly and the holomorphic tension gets no quantum corrections.

Keywords: Nonperturbative Effects, Solitons Monopoles and Instantons, Confinement, Matrix Models.

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1. Introduction

In the last years important progress has been made in the study of strongly coupled dynamics of various supersymmetric gauge theories. In particular, the understanding of supersymmetric constraints, the study of solitonic objects and various kind of dualities, have led to many exact results.

The works of Seiberg and Witten [1, 2] were the starting point for the understanding of the exact low energy dynamics of $\mathcal{N}=2$ theories [3]. In particular they found an exact formula for the mass of the BPS saturated dyons:

$$M_{\mathrm{BPS}} = \sqrt{2}|an_e + a_D n_m|\,, \tag{1.1}$$

where n_e and n_m are respectively the electric and the magnetic charge.

In this paper we find something similar for vortices (the Abrikosov-Nielsen-Olesen flux tubes [4]) in $\mathcal{N}=2$ SQCD broken to $\mathcal{N}=1$ by a superpotential $W(\Phi)$. When there is a color-flavor locking some flavors become massless in the $\mathcal{N}=2$ theory. Due to the presence of the superpotential, these flavors condense and create a vortex solution. The tension of the vortex can be written as a BPS tension plus a non-BPS contribution

$$T = T_{\mathrm{BPS}} + T_{\mathrm{non\,BPS}}\,, \tag{1.2}$$

where

$$T_{\mathrm{BPS}} = 4\pi|\mathcal{T}| \tag{1.3}$$

and we call $\mathcal{T}$ the *holomorphic tension*. In a recent work [5] these vortices were studied in the semiclassical limit ($m \gg \Lambda$), where the tension is the classical one plus quantum corrections that depend on the dynamical scale Λ

$$\mathcal{T} = W'(m) + \mathcal{O}\left(\frac{\Lambda}{m}\right). \tag{1.4}$$

In the present paper we compute the quantum corrections to the holomorphic tension and our result, in the semiclassical limit, is a resummation of infinite instanton contributions. We are not able to compute the non-BPS contribution $T_{\mathrm{non\,BPS}}$.

The formula (1.1) has a deep relation with the central charge of the $\mathcal{N}=2$ superalgebra. When the mass saturates the BPS bound, half of the supersymmetries are unbroken and this prevents quantum corrections. Something similar happens with vortices in an $\mathcal{N}=1$ theory. As studied in [6], if Lorentz invariance is broken by a vortex configuration, one can introduce a central charge in the $\mathcal{N}=1$ superalgebra. This central charge is essentialy the holomorphic tension $\mathcal{T}$.

Recently the work of Dijkgraaf and Vafa [7] has opened the way to new discoveries about the $\mathcal{N}=1$ non perturbative dynamics. In particular in [8, 9, 10] the $\mathcal{N}=2$ SQCD broken to $\mathcal{N}=1$ by a superpotential has been studied. The tools developed in these works will be essential in this paper.

Thanks to a generalized version of the Konishi anomaly one can compute all the expectation values of the operators in the chiral ring of the theory. The generators of the chiral ring are the power expansion in z of some quantities that are usually denoted $T(z)$, $R(z)$ and $M(z)$. These are differential forms on the Riemann surface $\Sigma_{\mathcal{N}=1}$ defined by the equation

$$y_m{}^2 = W'^2(z) + f(z)\,, \tag{1.5}$$

where $f(z)$ is a polynomial that depends on the vacuum. In the semiclassical limit, where $m \gg \Lambda$, the low energy degrees of freedom are the color locked quarks and so we expect the tension to be given by their condensate

$$\mathcal{T} = -Q\widetilde{Q}\,. \tag{1.6}$$

This condensate belongs to the chiral ring and can be computed exactly by the residue

$$\widetilde{Q}Q = \frac{1}{2\pi i}\oint_{\infty} M(z)dz = -\frac{1}{2}\Big(W' + \sqrt{W'^2 + f}\Big)\Big|_{z=m} \tag{1.7}$$

where m is the bare mass of the locked flavor.

Even if the generalized anomaly gives the condensate $Q\widetilde{Q}$ for every value of m, only in the semiclassical region these are the low energy degrees of freedom that create the vortex. Thus we expect that formula (1.6) will get other corrections and to get them we perform a computation in the strong coupling regime. The strong coupling is a regime in which the superpotential can be considered a small perturbation to the $\mathcal{N}=2$ theory. The following factorization equations give the relation between the Seiberg-Witten curve $\Sigma_{\mathcal{N}=2}$ and the curve $\Sigma_{\mathcal{N}=1}$:

$$y^2 = P_{N_c}(z)^2 - \Lambda^{2N_c-N_f}\prod_{I=1}^{N_f}(z-m_I) = F_{2n}(z)H_{N_c-n}(z)^2 \tag{1.8}$$

and

$$y_m{}^2 = W'(z)^2 + f(z) = g_k{}^2 F_{2n}(z)Q_{k-n}(z)^2\,. \tag{1.9}$$

This factorization has the following interpretation. The $\mathcal{N}=2$ low energy has N_c U(1) factors and $N_c - n$ of them are coupled to massless particles (this is seen in (1.8) by the $N_c - n$ double roots). One of these double roots is $\widetilde{m}$, that in the semiclassical limit becomes the bare mass of the locked flavor m. The computation of the tension gives

$$\mathcal{T} = \sqrt{W'^2 + f}\Big|_{z=\widetilde{m}}\,. \tag{1.10}$$

This is the correct result for the holomorphic tension valid for every m. The result (1.6), as expected, doesn't keep into account all the corrections (note that the last is evaluated at the bare mass m and not $\widetilde{m}$).

The tension of vortices, contrary to (1.1), has a non-BPS contribution that is already present in the simplest case: classical $\mathcal{N}=2$ SQED broken with a superpotential. As noted by many authors [11, 12], when the second derivative of the superpotential, W'', is not zero the tension is not the BPS one. In this case it is possible to ask when the corrections are small with respect to the holomorphic tension, and also if the confinement is of type I or type II.

A particularly strong result is obtained when the superpotential is linear. The non-BPS contribution vanishes for every value of m and the holomorphic tension is $4\pi|W'|$ without quantum corrections.

The paper is organized as follows. We start our analysis in section 2 by reviewing some properties of the vortices that arise in $\mathcal{N}=2$ SQED broken to $\mathcal{N}=1$ by a superpotential. This analysis is completely classical. We compute the holomorphic tension and see that the non-BPS contribution is already present at the classical level. The BPS tension is related to the central charge in the presence of a vortex configuration.

In section 3 we analyze the $\mathcal{N}=2$ $\mathrm{U}(N_c)$ gauge theory with N_f flavors, broken to $\mathcal{N}=1$ by a superpotential. The case $N_f < 2N_c$ is considered, so that the theory is asymptotically free. First we analyze the vortices classically where the tension is $T = 4\pi W'(m)$. Then we review the exact results of [8, 10] regarding the chiral ring of the theory. After this we are able to give a first approximation to the tension in the semiclassical region.

Section 4 is devoted to the computation in the strong coupling regime. The strong coupling is a regime in which the superpotential can be considered a small perturbation to the $\mathcal{N}=2$ theory. So one can use the exact results on the $\mathcal{N}=2$ low energy action and then add the effective superpotential. First we review the Seiberg-Witten curve that describes the low energy $\mathcal{N}=2$ dynamics. Then we recall the results concerning the factorization that relates the $\mathcal{N}=2$ curve to the $\mathcal{N}=1$ curve. Finally we perform the computation of the vortex tension and the result will be (1.10).

In section 5 we study the limit in which the non-BPS contribution can be neglected. We are able to give a condition of validity in the weak coupling regime. Section 6 is devoted to conclusion and discussion. In A we give some details for the calculation of the central charge in $\mathcal{N}=1$ SQED. In B we relate our conventions with the ones of [10].

2. Vortices in $\mathcal{N}=2$ SQED

The building block of the present work is $\mathcal{N}=2$ SQED. In this section we study this theory without considering quantum corrections. The U(1) gauge multiplied is composed by the superfields W_α and Φ, while the matter superfields are Q of charge $+1$ and $\widetilde{Q}$ of charge -1. $\mathcal{N}=2$ is broken to $\mathcal{N}=1$ by means of a superpotential that is a holomorphic function of Φ. The lagrangian is the following:

$$\mathcal{L} = \int d^2\theta \frac{1}{4e^2} W^\alpha W_\alpha + \text{h.c.} + \int d^2\theta d^2\bar{\theta}\left(\frac{1}{e^2}\Phi^\dagger\Phi + Q^\dagger e^V Q + \widetilde{Q}^\dagger e^{-V}\widetilde{Q}\right) + \\ + \int d^2\theta\,\sqrt{2}(\widetilde{Q}\Phi Q - m\widetilde{Q}Q + W(\Phi)) + \text{h.c.} \tag{2.1}$$

The potential for the scalar fields is

$$V = 2|(\phi - m)q|^2 + 2|(\phi - m)\widetilde{q}|^2 + 2e^2|\widetilde{q}q + W'(\phi)|^2 + \frac{e^2}{2}(|q|^2 - |\widetilde{q}|^2)^2\,, \tag{2.2}$$

and the vacuum solution is:

$$\phi = m\,, \qquad |q| = |\widetilde{q}|\,, \qquad \widetilde{q}q = -W'(m)\,. \tag{2.3}$$

The gauge group U(1) is completely broken by the quark condensate, so the theory admits a vortex configuration solution that belongs to the homotopy group $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$.

2.1 BPS solution

The homotopy consideration assures us that the vortex solution exists and is stable but in general it is difficult to find it. Let's start considering the BPS limit:

$$\phi = m\,, \qquad \widetilde{q} = -q^{\dagger}\frac{W'}{|W'|}\,. \tag{2.4}$$

Inserting the latter in the potential (2.2) one obtains $V(q) = 2e^2(|q|^2 - |W'|)^2$. Now the kinetic term of q has a factor 2, because it comes from q and $\widetilde{q}$, and to restore the proper normalization one has to rescale the field $q \to q/\sqrt{2}$. Thus the lagrangian becomes the usual one in the BPS limit [13]:

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - (D_\mu q)^\dagger(D^\mu q) - \frac{e^2}{2}(|q|^2 - 2|W'|)^2\,. \tag{2.5}$$

We orient the vortex in the $\hat{z}$ direction, so the tension is the integral of the energy density in $dx\,dy$, then, using the Bogomoln'y trick, the tension can be written as a sum of quadratic terms plus a boundary term:

$$\begin{aligned} T = &\int d^2x \frac{1}{2}|D_k q + i\epsilon_{kl}D_l q|^2 + \left(\frac{1}{2e}F_{kl} + \frac{e}{2}(|q|^2 - 2|W'|)\epsilon_{kl}\right)^2 + \\ &+ \oint d\vec{x}\cdot(2\vec{A}|W'| - iq^\dagger\vec{D}q)\,. \end{aligned} \tag{2.6}$$

A vortex of finite energy must be an element $[n] \in \pi_1(\mathrm{U}(1))$

$$q = e^{in\theta}\sqrt{2|W'|}\,q_n(r)\,, \qquad A_k = -n\epsilon_{kl}\frac{r_l}{r^2}\,f_n(r)\,, \tag{2.7}$$

where the profile functions satisfy these boundary conditions: $q_n(\infty), f_n(\infty) = 1$ and $q_n(0), f_n(0) = 0$. The q field is chosen so that it winds n times at infinity and the A_k field is chosen so that the covariant derivative $D_k q$ vanishes at infinity. The solution of the equation of motion, that give the profile functions, is the one that minimizes the tension (2.6), so we must put to zero the quadratic terms and the tension is given by the boundary term, that comes out to be proportional to the flux of the magnetic field:

$$T = 4\pi|\mathcal{T}|\,, \qquad \mathcal{T} = nW'(m)\,. \tag{2.8}$$

The BPS equations of motion are first order equations obtained by setting to zero the quadratic terms in the tension. Rescaling to a dimensionless length $\rho = e\sqrt{|W'|}r$ the equations for the profile functions are::

$$\begin{aligned} \frac{n}{\rho}\frac{df_{(n)}}{\rho} + {q_{(n)}}^2 - 1 &= 0\,, \\ \rho\frac{dq_{(n)}}{d\rho} + n(f_{(n)} - 1)q_{(n)} &= 0\,. \end{aligned} \tag{2.9}$$

This means that the radius of the vortex is $R_v \sim 1/e\sqrt{|W'|}$.

Now we discuss an important point regarding the BPS limit (2.4). This is only approximate and do not satisfy the complete set of the equations of motion and the simplest way to see this is to look at the equation of motion for ϕ:

$$\frac{\Box\phi}{e^2} = 2(\phi - m)(|q|^2 - |\tilde{q}|^2) + 2e^2 W''(\phi)^\dagger(\tilde{q}q + W'(\phi))\,. \tag{2.10}$$

With (2.4) one would obtain $0 = e^2 W''(m)^\dagger(|q|^2 - |W'(m)|)$ that is false because inside the radius of the vortex $|q|^2 - |W'(m)|$ is different from zero. The equation of motion is satisfied only if $W''(m) = 0$. Naively we can say that the BPS limit, even if not exact, could be a good approximation if $e \ll 1$. In section 5 we will give a more detailed analysis of the conditions in which the BPS limit is a good approximation.

2.2 Central charge

In A we study how the $\mathcal{N} = 1$ supersymmetric algebra is modified by the presence of a vortex configuration that makes possible the existence of a central charge. As explained in A, the central charge implies a BPS bound for the tension of the vortex

$$T_{\rm BPS} = 2r \oint d\vec{x} \cdot \vec{A}\,, \tag{2.11}$$

where r is the coefficient of the Fayet-Iliopulos term in the lagrangian. When this bound is saturated half of the supersymmetries are unbroken.

Now we want to apply the results of A to our theory: $\mathcal{N} = 2$ SQED broken to $\mathcal{N} = 1$ by a superpotential. A problem immediately arises: the central charge is zero because there is no FI term. But here there are two supersymmetries, so the $\mathrm{SU}(2)_R$ R-symmetry can be used to get some informations about the central charge. What we are going to do is to perform a $\mathrm{SU}(2)_R$ rotation to bring our theory in a form where there is no superpotential but there is a Fayet-Iliopoulos term [5, 12]. When both the superpotential and the FI term are present the potential is

$$V = 2e^2|\tilde{q}q + W'(m)|^2 + \frac{e^2}{2}(|q|^2 - |\tilde{q}|^2 - 2r)^2\,, \tag{2.12}$$

and this can be written in a $\mathrm{SU}(2)_R$ invariant form:

$$V = e^2 \mathrm{Tr}_2\left(q^{\dagger\alpha} q_\beta - \frac{1}{2}\delta^\alpha{}_\beta q^{\dagger\gamma} q_\gamma - \xi_a(\sigma_a)^\alpha{}_\beta\right)^2, \tag{2.13}$$

$$-\xi_1 + i\xi_2 = W'(m)\,, \qquad \xi_3 = r\,, \tag{2.14}$$

where q^α is the $\mathrm{SU}(2)_R$ doublet $(q, \tilde{q}^\dagger)$. Thus $(\mathrm{Re}\,W'(m), \mathrm{Im}\,W'(m), r)$ is a triplet of the SU(2) R-symmetry and, in our case, we may rotate the superpotential away leaving only an FI term whose coefficient is

$$r = |W'(m)|\,. \tag{2.15}$$

Using (2.11) we get the BPS tension

$$T_{\rm BPS} = 4\pi|W'(m)|\,, \tag{2.16}$$

and we recover (2.8).

3. Vortices in $\mathrm{U}(N_c)$ theory with N_f flavors: semiclassical limit

Here we come to the main subject of this article: the vortices in the $\mathcal{N}=2$ $\mathrm{U}(N_c)$ gauge theory with N_f flavors broken to $\mathcal{N}=1$ by means of a superpotential. We consider the case $N_f < 2N_c$ so that the theory is asymptotically free. To set the conventions we write the lagrangian of the theory:[1]

$$\begin{aligned}\mathcal{L} = &\int d^2\theta\, \frac{1}{2e^2}\mathrm{Tr}_{N_c}\,(W^\alpha W_\alpha) + \text{h.c.} + \int d^2\theta d^2\bar{\theta}\frac{2}{e^2}\mathrm{Tr}_{N_c}\,(\Phi^\dagger e^V \Phi e^{-V}) + \\ &+\int d^2\theta d^2\bar{\theta}\sum_{I=1}^{N_f}(Q_I{}^\dagger e^V Q^I + \widetilde{Q}_I e^{-V}\widetilde{Q}^{\dagger I}) + \int d^2\theta \sum_{I=1}^{N_f}\sqrt{2}(\widetilde{Q}_I\Phi Q^I - m_I\widetilde{Q}_I Q^I) + \\ &+\sqrt{2}\mathrm{Tr}_{N_c}\,W(\Phi) + \text{h.c.}\,, \end{aligned} \tag{3.1}$$

where

$$W(z) = \sum_{j=0}^{k}\frac{g_j}{j+1}z^{j+1}\,, \qquad W'(z) = g_k\prod_{j=1}^{k}(z-a_j)\,. \tag{3.2}$$

Here, as in [10], we consider the case in which all the roots a_j and all the masses m_I are different.

3.1 Classical analysis

Now we consider the weak coupling regime (for a detailed analysis see [5]). The diagonal elements of the adjoint field are equal to a flavor mass m_I (the color-flavor locking) or to a root of W':

$$\langle\phi\rangle = \begin{pmatrix} \ddots & & & & \\ & m_I\mathbf{1}_{r_I} & & & \\ & & \ddots & & \\ & & & a_j\mathbf{1}_{N_j} & \\ & & & & \ddots \end{pmatrix}, \qquad \sum_{j=1}^{n}N_j + \sum_{I=1}^{N_f} r_I = N_c\,. \tag{3.3}$$

The gauge group is classically broken to $\prod_{j=1}^n \mathrm{U}(N_j)$, where n is less than or equal to the number of the roots W' and r_I is less than or equal to the number of flavors with mass m_I. In this case, where all the roots are distinct, there are only two possibilities: $r_I = 0, 1$. When $r_I = 1$ the flavors Q^I_a and $\widetilde{Q}^a_I$ have zero mass and at low energy one has $\mathcal{N}=2$ SQED broken to $\mathcal{N}=1$ by a superpotential, that is the same theory studied in 2.1. Thus the results of section 2 can be applied and so the theory develops a vortex of tension

$$T = 4\pi|W'(m)|\,. \tag{3.4}$$

Note that this analysis has been carried out by considering the lagrangian (3.1) classically. When $m \gg \Lambda$ we can trust this results as a good approximation because the theory

[1] Concerning the generators of the group, we use the $N_c\times N_c$ matrices $T^{a\,\dagger} = T^a$ with normalization $\mathrm{Tr}_{N_c}\,(T^aT^b) = \delta^{ab}/2$.

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is at weak coupling. In the following we are going to compute the quantum corrections to the tension. To do this we need first a brief review of the Cachazo-Douglas-Seiberg-Witten solution.

3.2 CDSW solution

Here we recall the results of [8, 9, 10] (see [14] for a review), focusing on the points that we need to compute the corrections in subsection 3.3. Consider the following operators:

$$T(z) = \mathrm{Tr}\,\frac{1}{z-\Phi}\,, \qquad R(z) = -\frac{1}{16\sqrt{2}\pi^2}\mathrm{Tr}\,\frac{W^\alpha W_\alpha}{z-\Phi}\,, \tag{3.5}$$

$$M_I(z) = \widetilde{Q}_I\frac{1}{z-\Phi}Q_I\,. \tag{3.6}$$

Taking the coefficients of the power expansion in z one obtains all the generators of the chiral ring of the theory. The generalized Konishi anomalies [8, 10, 15] provides a solution for the chiral ring. The anomalies that we need are the following:

$$\begin{aligned} [W'(z)R(z)]_- &= R(z)^2\,, \\ [M_I(z)(z-m_I)]_- &= R(z)\,. \end{aligned} \tag{3.7}$$

The solution of the first equation is

$$2R(z) = W'(z) - \sqrt{W'(z)^2 + f(z)}\,, \tag{3.8}$$

where $f(z)$ is a polynomial of degree $k-1$ that depends on the vacuum. By (3.8) we are naturally led to consider the Riemann surface $\Sigma_{\mathcal{N}=1}$ defined by the equation

$$y^2 = W'(z)^2 + f(z)\,. \tag{3.9}$$

This is a double sheeted cover of the complex plane on which $R(z)$ is uniquely defined. We call q_I and $\widetilde{q}_I$ the two points of $\Sigma_{\mathcal{N}=1}$ with the same coordinate $z = m_I$. When $r_I = 0$ there is no color-flavor locking and $M_I(z)$ must be regular in q_I:

$$M_I(z) = \frac{R(z)}{z-m_I} - \frac{R(q_I)}{z-m_I}\,. \tag{3.10}$$

When $r_I = 1$ one can find the solution by continuously deforming the theory, in such a way that the pole passes from the second to the first sheet. The solution is:

$$M_I(z) = \frac{R(z)}{z-m_I} - \frac{W'(m_I) - R(q_I)}{z-m_I}\,. \tag{3.11}$$

In the semiclassical limit, where $f(z) \to 0$, the cuts of $\Sigma_{\mathcal{N}=1}$ are closed and the Riemann surface becomes simply the Riemann sphere $\hat{\mathbb{C}}$ with punctures at a_j and m_I. The gaugino condensate is $R(z) = 0$, as it should be, and the quark condensate is

$$M_I(z) = -\frac{W'(m_I)}{z-m_I}\,. \tag{3.12}$$

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3.3 The holomorphic tension: a first approximation

In the semiclassical limit ($m \gg \Lambda$) the quarks Q_I and $\widetilde{Q}_I$ are the low energy degrees of freedom that, upon breaking to $\mathcal{N}=1$, condense and create the vortex. So we expect that the holomorphic tension is given by their condensate as in (2.1)

$$\mathcal{T}_I = -\widetilde{Q}_I Q_I \,. \tag{3.13}$$

This condensate belongs to the chiral ring and can be computed using the results of subsection 3.2. $\widetilde{Q}_I Q_I$ is given by the $1/z$ pole of the generator $M_I(z)$

$$\widetilde{Q}_I Q_I = \frac{1}{2\pi i}\oint_\infty M_I(z)dz = -\frac{1}{2}\Big(W' + \sqrt{W'^2 + f}\Big)\Big|_{z=m_I}\,, \tag{3.14}$$

This result is only a first approximation to the holomorphic tension because (3.13) and (3.14) would be exact only if the low energy superpotential was:

$$\mathcal{W}_{\text{low}} = \sqrt{2}(\widetilde{Q}_I(\Phi - m_I)Q_I + \mathcal{W}_{\text{eff}}(\Phi))\,. \tag{3.15}$$

On the other hand, in the low energy lagrangian there could also be terms like $\mathrm{Tr}_{N_c}\,\Phi^\alpha$, $\widetilde{Q}_I\Phi^\beta Q_I$ or products of them. In particular terms such as $\widetilde{Q}_I P(\Phi) Q_I$, where $P(z)$ is a polynomial, will have the effect of shifting the mass m_I. The right result will be computed in the next section using the factorized curves in the strong coupling regime (see (4.26)).

4. Strong coupling computation of $\mathcal{T}$

Here we compute the holomorphic tension at strong coupling, where W is a small perturbation of the $\mathcal{N}=2$ theory. In this regime we can take the low energy $\mathcal{N}=2$ theory and add the effective superpotential generated by W.

4.1 Low energy $\mathcal{N}=2$

Here we recall the results about the low energy dynamics of the $\mathcal{N}=2$ theory (see [16] for the conventions and [17] for a review).

First we consider $\mathrm{SU}(N_c)$ with N_f flavors. At low energy one has N_c-1 $\mathrm{U}(1)$ gauge multiplets and we call their scalar components a_i, where $i=1,\ldots,N_c-1$. The moduli space is a N_c-1 dimensional complex manifold $\mathcal{M}_{\mathrm{SU}(N_c)}$, parametrized by the gauge invariant coordinates

$$u_j = \frac{1}{j}\langle \mathrm{Tr}\,\phi^j\rangle\,, \qquad j = 2,\ldots,N_c\,. \tag{4.1}$$

The informations concerning the low energy dynamics are encoded in the Riemann surface $\Sigma_{\mathcal{N}=2}$ defined by

$$y^2 = P_{N_c}(z)^2 - \Lambda^{2N_c-N_f}\prod_{I=1}^{N_f}(z-m_I)\,, \tag{4.2}$$

where $P_{N_c}(z) = \det(z - \phi)$ can be written in power series of z

$$\begin{aligned} P_{N_c}(z) &= \sum_{k=0}^{N_c} s_k z^{N_c - k} \,, \\ s_0 &= 1 \,, \qquad s_1 = 0 \,, \qquad s_k = (-)^k \sum_{i_1 < \ldots < i_k} \phi_{i_1} \ldots \phi_{i_k} \,. \end{aligned} \tag{4.3}$$

Being $\Sigma_{\mathcal{N}=2}$ a genus $N_c - 1$ Riemann surface, we can choose $N_c - 1$ independents holomorphic differentials:[2]

$$\lambda_j \propto \frac{z^{N_c - j} dz}{y} \,, \qquad j = 2, \ldots, N_c \,. \tag{4.4}$$

Each a_i corresponds to an α_i cycle on $\Sigma_{\mathcal{N}=2}$, while its dual a_{Dj} corresponds to a β_j cycle chosen in such a way that the intersection is $< \alpha_i, \beta_j >= \delta_{ij}$. The solution is given by the period integrals

$$\frac{\partial a_i}{\partial s_j} = \oint_{\alpha_i} \lambda_j \,, \qquad \frac{\partial a_{Di}}{\partial s_j} = \oint_{\beta_i} \lambda_j \,. \tag{4.5}$$

The relation between u_j and s_k will be important for us because the solution (4.5) gives $\partial a_i / \partial s_j$ but, to calculate the tension, we will need $\partial a_i / \partial u_j$. These relations can be encoded in a single one [9]:

$$P_{N_c}(z) = z^{N_c} \exp \left(- \sum_{j=1}^{\infty} \frac{u_j}{z^j} \right)_+ \,, \tag{4.6}$$

where by $(\;)_+$ we mean that we discard the negative power expansion.

Now we study the $\mathrm{U}(N_c)$ theory with N_f flavors and we are going to see that the solution can be easily incorporated in the previous ones, with few modifications. The low energy theory has one more $\mathrm{U}(1)$ factor that comes from the decomposition $\mathrm{U}(N_c) = \mathrm{U}(1) \times \mathrm{SU}(N_c)$ and we denote its scalar component with a_{N_c}. This factor has no strong dynamics: in the $N_f = 0$ case it is completely free, while in the $N_f \neq 0$ case it is infrared free. The moduli space $\mathcal{M}_{\mathrm{U}(N_c)}$ has one dimension more and is parametrized by

$$u_j = \frac{1}{j} \langle \mathrm{Tr}\, \phi^j \rangle \,, \qquad j = 1, \ldots, N_c \,. \tag{4.7}$$

The Riemann surface is the same given in (4.2), but here ϕ can have non zero trace and $\Sigma_{\mathcal{N}=2}$ depends also on the modulus u_1. To complete our task we must find the cycle α_{N_c} that corresponds to a_{N_c} and the differential λ_1 that corresponds to s_1. The cycle α_{N_c} is the one that encircles all the cuts in the z plane. Note that this is a trivial cycle and only a meromorphic differential can be different from zero when it is integrated around it. The differential that corresponds to $s_1 = -u_1$ is

$$\lambda_1 \propto \frac{z^{N_c - 1} dz}{y} \tag{4.8}$$

and is meromorphic because it has a pole at ∞. With these modifications the solution is encoded in (4.5).

[2]The normalization will be fixed imposing the correct semiclassical result.

4.2 Breaking to $\mathcal{N} = 1$

Now we break $\mathcal{N} = 2$ to $\mathcal{N} = 1$ by a superpotential. This breaking leaves only a discrete number of vacua. If the low energy is $\mathrm{U}(1)^n$, than $N_c - n$ gauge factors are broken by the condensate of a charged field. These charged fields must be massless in the $\mathcal{N} = 2$ theory and so the Riemann surface must have $N_c - n$ degenerate branch cuts

$$y^2 = P_{N_c}(z)^2 - \Lambda^{2N_c - N_f} \prod_{I=1}^{N_f} (z - m_I) = F_{2n}(z) H_{N_c - n}(z)^2 \,. \tag{4.9}$$

The connection with the $\mathcal{N} = 1$ curve (3.9) is given by the following factorization [9, 10, 18, 19]

$$y_m{}^2 = W'(z)^2 + f(z) = g_k{}^2 F_{2n}(z) Q_{k-n}(z)^2 \,. \tag{4.10}$$

The above factorization provides all the informations we need: from the knowledge of W one can obtain the vacua that survive and the correspondent $\mathcal{N} = 1$ curve. As in the classical case, two conditions must be satisfied: $n \leq k$ and $n \leq N_c$. When $k = n$ the factorization is simply

$$y^2 = P_{N_c}(z)^2 - \Lambda^{2N_c - N_f} \prod_{I=1}^{N_f} (z - m_I) = \frac{1}{g_k{}^2} (W'^2(z) + f(z)) H_{N_c - n}(z)^2 \,. \tag{4.11}$$

Consider the case where $N_c - n$ roots of (4.2) collide and, as we said before, $N_c - n$ of the U(1) gauge factors will have massless charged field that we call $Q_r, \widetilde{Q}_r$, with $r = 1, \ldots, N_c - n$. The low energy superpotential is the $\mathcal{N} = 2$ one plus a holomorphic function of the chiral superfields A_i

$$\mathcal{W}_{\text{low}} = \sqrt{2} \Big(\sum_{r=1}^{N_c - n} \widetilde{Q}_r A_r Q_r + \mathcal{W}_{\text{eff}}(A_1, \ldots, A_{N_c}) \Big) \,. \tag{4.12}$$

By means of holomorphic arguments [1] one can show that, when the tree superpotential is (3.2), the effective superpotential is

$$\mathcal{W}_{\text{eff}} = \sum_{j=1}^{k} g_j u_{j+1}(A_1, \ldots, A_{N_c}) \,, \tag{4.13}$$

where the $u_{j+1}(A_1, \ldots, A_{N_c})$ are given implicitly by the solution (4.5). For our proof we need to consider only the F_A terms of the potential:

$$F_{A_r} = 2 e_r{}^2 \left| \tilde{q}_r q_r + \frac{\partial \mathcal{W}_{\text{eff}}}{\partial a_r} \right|^2 , \qquad r = 1, \ldots, N_c - n \,, \tag{4.14}$$

$$F_{A_s} = 2 e_s{}^2 \left| \frac{\partial \mathcal{W}_{\text{eff}}}{\partial a_s} \right|^2 , \qquad s = N_c - n + 1, \ldots, N_c \,. \tag{4.15}$$

The first one gives the holomorphic tension of the r-vortex, while the second gives a stationary condition:

$$\mathcal{T}_r = -\tilde{q}_r q_r = \frac{\partial \mathcal{W}_{\text{eff}}}{\partial a_r} \,, \tag{4.16}$$

$$0 = \frac{\mathcal{W}_{\text{eff}}}{\partial a_s} \,. \tag{4.17}$$

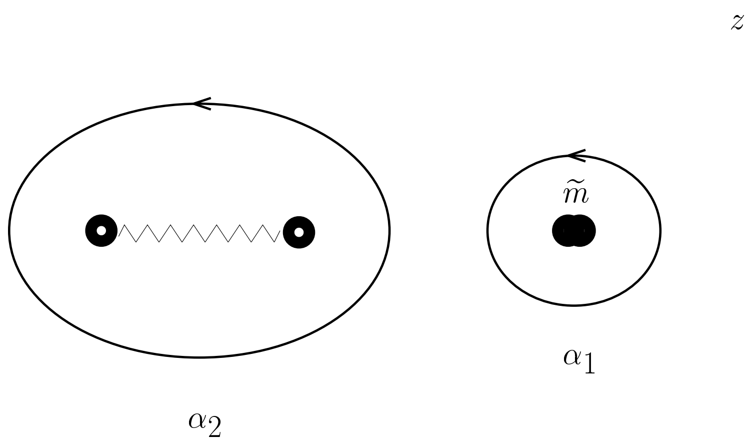

Figure 1: Cycles in U(2) theory.

4.3 Computation of $\mathcal{T}$

We are now ready to perform the computation of the holomorphic tension. We will start by the simplest case, then we will consider step by step more general cases.

4.3.1 The case: $n = k = N_c - 1$.

The simplest case of this category is $N_c = 2$ with $k = n = 1$ where the superpotential is

$$W(z) = g_0 z + \frac{g_1}{2} z^2 \,. \tag{4.18}$$

The Riemann surface $\Sigma_{\mathcal{N}=2}$ has two cuts (see figure 1), and the first one is shrank to a point $z = \widetilde{m}$. We denote by α_1 and α_2 the cycles encircling the two cuts. The factorization (4.11) gives

$$\Sigma_{\mathcal{N}=2}: \quad y^2 = \frac{1}{{g_1}^2}(W'^2 + f)(z - \widetilde{m})^2 \,, \tag{4.19}$$

while the integrals around the cycle α_1 become simply the residues around the point $\widetilde{m}$, for example

$$\frac{1}{2\pi i}\oint_{\alpha_1} \frac{dz}{y} = \frac{g_1}{\sqrt{W'^2 + f}} \,. \tag{4.20}$$

In this case the relation between s and u is:

$$s_1 = -u_1 \,, \qquad s_2 = -u_2 + \frac{{u_1}^2}{2} \,. \tag{4.21}$$

For our proof we will need to calculate only

$$\frac{\partial a_1}{\partial u_2} = \frac{\partial a_1}{\partial s_1}\frac{\partial s_1}{\partial u_2} + \frac{\partial a_1}{\partial s_2}\frac{\partial s_2}{\partial u_2} = -\frac{\partial a_1}{\partial s_2} \,. \tag{4.22}$$

First, we observe that the solution (4.5) and the residue (4.20) give:

$$\frac{\partial a_1}{\partial s_2} = -\frac{g_1}{\sqrt{W'^2 + f}} \,. \tag{4.23}$$

Then, writing the equations (4.16) and (4.17) in a matrix form and left multiplying by the inverse matrix, we get

$$\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} \partial a_1/\partial u_1 & \partial a_2/\partial u_1 \\ \partial a_1/\partial u_2 & \partial a_2/\partial u_2 \end{pmatrix} \begin{pmatrix} \mathcal{T} \\ 0 \end{pmatrix}. \tag{4.24}$$

The simple passage of multiplying by the inverse matrix has simplified a lot our work because now (4.24) is expressed as a function of $\partial a_i/\partial u_j$, known directly through (4.5). Furthermore only $\partial a_1/\partial u_{1,2}$, the ones obtained by an integral around the collided branch, are important because the others are multiplied by zero. Actually by (4.24) we need only the second equation

$$g_1 = \mathcal{T}\frac{\partial a_1}{\partial u_2}\,, \tag{4.25}$$

that, using (4.23), gives the holomorphic tension:

$$\mathcal{T} = \sqrt{W'^2 + f}\Big|_{z=\widetilde{m}}\,. \tag{4.26}$$

Let us see what happens in the general case $n = k = N_c - 1$. We can write a matrix equation like (4.24) where the couplings vector is

$$\mathbf{g} = (g_0, \ldots, g_{N_c-1}) \tag{4.27}$$

and the tension vector is

$$\mathbf{T} = (\mathcal{T}, 0, \ldots, 0)\,. \tag{4.28}$$

With these conventions the matrix equation that generalizes (4.24) becomes:

$$\mathbf{g} = \frac{\partial \mathbf{a}}{\partial \mathbf{u}}\mathbf{T}\,. \tag{4.29}$$

As in the simplest case examined before, where we needed only the second equation of (4.24), now we need only the last one among the equations contained in the matrix relation (4.29):

$$g_{N_c-1} = \mathcal{T}\frac{\partial a_1}{\partial u_{N_c}}\,. \tag{4.30}$$

The derivative can be easily calculated

$$\frac{\partial a_1}{\partial u_{N_c}} = -\frac{\partial a_1}{\partial s_{N_c}} = \frac{g_{N_c-1}}{\sqrt{W'^2 + f}} \tag{4.31}$$

and one still gets (4.26) for the holomorphic tension.

4.3.2 The case: $n = k < N_c - 1$.

The next step consists in letting more flavors to be locked. We still consider $k = n$, in such a way that the factorization is the simplest one, given in (4.11).

First we give the proof for the simplest example of this kind i.e. U(3) with two flavors of mass m_1 and m_2. The superpotential is

$$W(z) = g_0 z + \frac{g_1}{2}z^2\,, \tag{4.32}$$

while the factorization gives

$$\Sigma_{\mathcal{N}=2}: \quad y^2 = \frac{1}{{g_1}^2}(W'^2 + f)(z - \widetilde{m}_1)^2 (z - \widetilde{m}_2)^2 \,. \tag{4.33}$$

The relation between s and u in this case is:

$$s_1 = -u_1 \,, \qquad s_2 = -u_2 + \frac{{u_1}^2}{2} \,, \qquad s_3 = -u_3 + u_1 u_2 - \frac{{u_1}^3}{6} \,. \tag{4.34}$$

By using the same trick explained before, we write equations (4.16) and (4.17) in a matrix form and multiply by the inverse, obtaining

$$\begin{pmatrix} g_0 \\ g_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial a_1/\partial u_1 & \ldots & \partial a_3/\partial u_1 \\ \vdots & & \vdots \\ \partial a_1/\partial u_3 & \ldots & \partial a_3/\partial u_3 \end{pmatrix} \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \\ 0 \end{pmatrix} \,. \tag{4.35}$$

Now, as in the previous case, we need to calculate only the residues around $\widetilde{m}_1$ and $\widetilde{m}_2$. The last equation of (4.35) is enough for our proof:

$$0 = \mathcal{T}_1 \frac{1}{\sqrt{W'^2 + f}\Big|_{\widetilde{m}_1} (\widetilde{m}_1 - \widetilde{m}_2)} + \mathcal{T}_2 \frac{1}{\sqrt{W'^2 + f}\Big|_{\widetilde{m}_2} (\widetilde{m}_2 - \widetilde{m}_1)} \,. \tag{4.36}$$

If we impose that $\mathcal{T}_1$ does not depend on $\widetilde{m}_2$ and $\mathcal{T}_2$ does not depend on $\widetilde{m}_1$,[3] the solution is unique:

$$\mathcal{T}_1 = \sqrt{W'^2 + f}\Big|_{\widetilde{m}_1} \,, \qquad \mathcal{T}_2 = \sqrt{W'^2 + f}\Big|_{\widetilde{m}_2} \,. \tag{4.37}$$

The requirement that $\mathcal{T}_I$ depends only on its mass $\widetilde{m}_I$ has brought a great simplification because only the last of the matrix equation is necessary to find the solution. Actually in this simple case one can verify that the requirement is indeed true by using also the second equation of (4.35). Using

$$\frac{\partial a}{\partial u_2} = -\frac{\partial a}{\partial s_2} + u_1 \frac{\partial a}{\partial s_3} \,, \tag{4.38}$$

the second equation of (4.35) leads to another independent equation

$$1 = \mathcal{T}_1 \frac{\widetilde{m}_1}{\sqrt{W'^2 + f}\Big|_{\widetilde{m}_1} (\widetilde{m}_1 - \widetilde{m}_2)} + \mathcal{T}_2 \frac{\widetilde{m}_2}{\sqrt{W'^2 + f}\Big|_{\widetilde{m}_2} (\widetilde{m}_2 - \widetilde{m}_1)} \,. \tag{4.39}$$

This equation together with (4.36) are enough to establish the solution (4.37) and in particular to verify that $\mathcal{T}_I$ depends only on $\widetilde{m}_I$.

We now consider the general case $n = k < N_c - 1$. The factorization of the curve gives

$$\Sigma_{\mathcal{N}=2}: \quad y^2 = \frac{1}{{g_n}^2}(W'^2 + f)(z - \widetilde{m}_1)^2 \ldots (z - \widetilde{m}_{N_c - n})^2 \,, \tag{4.40}$$

[3]We mean that $\mathcal{T}_I$ doesn't depend on $\widetilde{m}_{J \neq I}$ if we vary $\widetilde{m}_J$ keeping fixed $W(z)$ and $f(z)$.

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while the vectors of the coupling and of the tension are:

$$\begin{aligned}\mathbf{g} &= (g_0, \dots, g_n, 0, \dots, 0)\,,\\ \mathbf{T} &= (\mathcal{T}_1, \dots, \mathcal{T}_{N_c-n}, 0, \dots, 0)\,.\end{aligned} \tag{4.41}$$

The last $N_c - n$ equations of (4.29) are:

$$\begin{aligned}1 &= \sum_J \mathcal{T}_J \frac{\widetilde{m}_J^{\,N_c-n-1}}{\sqrt{W'^2+f}\Big|_{\widetilde{m}_J} \prod_{I\neq J}(\widetilde{m}_J - \widetilde{m}_I)}\,,\\ 0 &= \sum_J \mathcal{T}_J \frac{\widetilde{m}_J^{\,N_c-n-2}}{\sqrt{W'^2+f}\Big|_{\widetilde{m}_J} \prod_{I\neq J}(\widetilde{m}_J - \widetilde{m}_I)}\,,\\ &\vdots\\ 0 &= \sum_J \mathcal{T}_J \frac{1}{\sqrt{W'^2+f}\Big|_{\widetilde{m}_J} \prod_{I\neq J}(\widetilde{m}_J - \widetilde{m}_I)}\,.\end{aligned} \tag{4.42}$$

If we put the desired result $\mathcal{T}_J = \sqrt{W'^2+f}\Big|_{\widetilde{m}_J}$, we obtain the following terms with $r = 1 \dots N_c - n$:

$$\sum_J \frac{\widetilde{m}_J^{\,r}}{\prod_{I\neq J}(\widetilde{m}_J - \widetilde{m}_I)} = \frac{\sum_J (-)^J \widetilde{m}_J^{\,r} \prod_{I<K;\,I,K\neq J}(\widetilde{m}_I - \widetilde{m}_K)}{\prod_{I<K}(\widetilde{m}_I - \widetilde{m}_K)}\,. \tag{4.43}$$

It's easy to see that the numerator is a multiple of the denominator because if some $\widetilde{m}_I$ are equal to some of the $\widetilde{m}_K$ the numerator vanishes. On the other hand if $r < N_c - n$ the proportionality constant must vanish because its power is less that the power of the denominator. When $r = N_c - n$, the fraction is equal to one. Summarizing the results we obtain:

$$\sum_J \frac{\widetilde{m}_J^{\,r}}{\prod_{I\neq J}(\widetilde{m}_J - \widetilde{m}_I)} = \begin{cases} 0 & r < N_c - n \\ 1 & r = N_c - n \end{cases} \tag{4.44}$$

and the equations (4.42) are satisfied.

4.3.3 The case: $n < k \leq N_c$.

Here the case $n < k$ is considered with the factorization given by (4.9), (4.10). The simplest example of this category is U(3) with two flavors and the following superpotential:

$$W(z) = g_0 z + \frac{g_1}{2} z^2 + \frac{g_2}{3} z^3\,. \tag{4.45}$$

If one denotes $Q_{k-n}(z) = z - \gamma$ in (4.10), then the factorization gives

$$\Sigma_{\mathcal{N}=2}: \qquad y^2 = \frac{1}{g_2{}^2}(W'^2 + f)\frac{(z-\widetilde{m}_1)^2(z-\widetilde{m}_2)^2}{(z-\gamma)^2}\,. \tag{4.46}$$

The matrix equation can be written as

$$\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \partial a_1/\partial u_1 & \dots & \partial a_3/\partial u_1 \\ \vdots & & \vdots \\ \partial a_1/\partial u_3 & \dots & \partial a_3/\partial u_3 \end{pmatrix} \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \\ 0 \end{pmatrix} \tag{4.47}$$

and the last equation is

$$1 = \mathcal{T}_1 \frac{\widetilde{m}_1 - \gamma}{\sqrt{W'^2 + f}\Big|_{\widetilde{m}_1} (\widetilde{m}_1 - \widetilde{m}_2)} + \mathcal{T}_2 \frac{\widetilde{m}_2 - \gamma}{\sqrt{W'^2 + f}\Big|_{\widetilde{m}_2} (\widetilde{m}_2 - \widetilde{m}_1)} . \tag{4.48}$$

If we impose that $\mathcal{T}_1$ depends only on m_1 and that $\mathcal{T}_2$ depends only on m_2, the unique solution of this equation is again (4.37).

In the general case $n < k \leq N_c$ the factorization is

$$\Sigma_{\mathcal{N}=2} : \quad y^2 = \frac{1}{g_k{}^2}(W'^2 + f)\frac{(z-\widetilde{m}_1)^2 \dots (z-\widetilde{m}_{N_c-n})^2}{(z-\gamma_1)^2 \dots (z-\gamma_{k-n})^2} \tag{4.49}$$

while the coupling and the tension vectors are:

$$\begin{aligned} \mathbf{g} &= (g_0, \dots, g_k, 0, \dots, 0)\,, \\ \mathbf{T} &= (\mathcal{T}_1, \dots, \mathcal{T}_{N_c-n}, 0, \dots, 0)\,. \end{aligned} \tag{4.50}$$

We use only the last one of the matrix equation and the condition that $\mathcal{T}_I$ depends only on $\widetilde{m}_I$. Two cases must be distinguished: the case $k < N_c$ where the last equation is

$$0 = \sum_J \mathcal{T}_J \frac{\prod_{q=1}^{k-n} (\widetilde{m}_J - \gamma_q)}{\sqrt{W'^2 + f}\Big|_{\widetilde{m}_J} \prod_{I\neq J}(\widetilde{m}_J - \widetilde{m}_I)} , \tag{4.51}$$

and the case $k = N_c$, where it is

$$1 = \sum_J \mathcal{T}_J \frac{\prod_{q=1}^{N_c-n} (\widetilde{m}_J - \gamma_q)}{\sqrt{W'^2 + f}\Big|_{\widetilde{m}_J} \prod_{I\neq J}(\widetilde{m}_J - \widetilde{m}_I)} . \tag{4.52}$$

Using (4.43) one can show that the solution is $\mathcal{T}_J = \sqrt{W'^2 + f}\Big|_{\widetilde{m}_J}$.

4.3.4 General case: $n \leq k$ and $n \leq N_c$.

Let us finally consider the most general case: $n \leq k$ and $n \leq N_c$. What is new here is that k can be greater than N_c and so to compute the tension we should evaluate also the derivatives $\partial u_{k>N_c}/\partial a$. Only the first N_c of the u_k are independents, and the expansion in negative powers of (4.6) gives the classical relations to obtain the $u_{k>N_c}$'s as functions of the previous ones. Some of these relations gets quantum corrections, so the brute force computation is difficult.

The assumption that $\mathcal{T}_I$ doesen't depends on $\widetilde{m}_{J\neq I}$ helps us to overcome this problem. Keeping $W(z)$ and $f(z)$ fixed we can add a color locked flavor of mass $\widetilde{m}_J$ bringing it from infinity. This procedure do not change k, n and, by our assumption, neither $\mathcal{T}_I$. The effect of this change is to obtain a theory with one more color and one more flavor

$$N_c\,, N_f \longrightarrow N_c + 1\,, N_f + 1\,. \tag{4.53}$$

We can repeat this procedure untill the new N_c is greater than k, and so the tension can be calculated as in the previous cases.

4.3.5 $\mathcal{T}_I$ doesn't depend on $\widetilde{m}_{J\neq I}$.

Our check was based on an assumption still not completely proved: the tension $\mathcal{T}_I$ do not change if we move $\widetilde{m}_{J\neq I}$ and keep fixed $W(z)$ and $f(z)$. We stress that this information should be contained in the full set of equations, but only for the simplest cases we were able to give a complete computation without using it.

Here, based on the classical limit and holomorphy, we give a valid argument in favor of this assumption. Holomorphy tells that, if a function doesn't depends on some parameter in some region, it doesn't depend on that parameter globally. In the classical limit the holomorphic tension is $W'(m_I)$. Suppose that there are holomorphic corrections that depends on $m_{J\neq I}$, they should be of two types: a positive power or a negative power of $m_{J\neq I}$. The first case, as example

$$\frac{\Lambda m_{J\neq I}}{m_I^{\ 2}}\,, \tag{4.54}$$

can be immediately excluded. If we bring $m_{J\neq I}$ to infinity this correction grows but the flavor should decouple from the theory. Also the second case, as example

$$\frac{\Lambda}{m_{J\neq I}}\,, \tag{4.55}$$

can be excluded. This correction should grows if we bring $m_{J\neq I}$ to zero and at some point dominate the classical tension $W'(m_I)$, but if we simultaneously bring m_I to infinity the classical tension should dominate. This contradictions exclude the possibility of holomorphic corrections to the I-tension that depends on $m_{J\neq I}$.

5. Limit of validity

The tension, as we said in subsection 2.1, has a non-BPS contribution that we are not able to compute. Here we consider in more detail this contribution and the limit in which it can be neglected because small with respect to the BPS tension.

5.1 Classical and quantum SQED

First consider SQED at a classical level. We said that the BPS vortex (2.4) do not satisfy the complete set of the equation of motion, so we write the tension from the lagrangian (2.1) without making any approximation

$$\begin{aligned} T = \int d^2x\, \frac{1}{4e^2}F_{kl}F_{kl} + \frac{1}{e^2}\partial_k\phi^\dagger\partial_k\phi + (D_kq)^\dagger(D_kq) + (D_k\widetilde{q})^\dagger(D_k\widetilde{q}) + \\ +2|(\phi-m)q|^2 + 2|(\phi-m)\widetilde{q}|^2 + 2e^2|\widetilde{q}q + W'(\phi)|^2 + \frac{e^2}{2}(|q|^2-|\widetilde{q}|^2)^2\,. \end{aligned} \tag{5.1}$$

We want to compute the first order correction to the BPS tension and, comparing it to the BPS tension, we get the condition in which the non-BPS contribution can be neglected. The stationary equations deriving from (5.1) are:

$$\begin{aligned} &\frac{\triangle\phi}{e^2} = 2(\phi-m)(|q|^2-|\widetilde{q}|^2) + 2e^2W''(\phi)^\dagger(\widetilde{q}q+W'(\phi))\,, \\ &\triangle q = \dots\,, \qquad \triangle\widetilde{q} = \dots\,, \qquad \partial_kF_{kl} = \dots \end{aligned} \tag{5.2}$$

To compute the first order correction we use the following technique. First we put in the right member of the equations of motion the BPS solution (2.4) and (2.7) and then we solve obtaining the solution corrected to the first order. The equations in the second row of (5.2) need no corrections, thus the only correction is given by $\phi = m + \phi_{(1)}$ and the equation in the first row

$$\triangle\phi_{(1)} = 2e^4 W''(m)^\dagger(\tilde{q}q + W'(m)) \,. \tag{5.3}$$

From this equation we are able to give an estimation for $\phi_{(1)}$. Outside the radius of the vortex R_v, $\phi_{(1)}$ is zero, while inside $\phi_{(1)} \sim e^4 W'' W' {R_v}^2$ (remember that from (2.9) $R_v \sim 1/e\sqrt{|W'|}$). The first order correction to the tension comes from three pieces: the first is the kinetic term of ϕ

$$\int d^2x \, \frac{1}{e^2} \partial_k {\phi_{(1)}}^\dagger \partial_k \phi_{(1)} \sim e^6 W''^2 W'^2 {R_v}^4 \sim e^2 W''^2 \,, \tag{5.4}$$

the second is the sum of the F_q and $F_{\tilde{q}}$ terms

$$\int d^2x \, 2|\phi_{(1)} q|^2 + 2|\phi_{(1)}\tilde{q}|^2 \sim e^8 W''^2 W'^3 {R_v}^6 \sim e^2 W''^2 \,. \tag{5.5}$$

and the last is the deformation of the F_ϕ term

$$\int d^2x \, 2e^2 \frac{\partial}{\partial\phi} |\tilde{q}q + W'(\phi)|^2 \phi_{(1)} \sim e^6 W''^2 W'^2 {R_v}^4 \sim e^2 W''^2 \,. \tag{5.6}$$

All these three corrections are of the same order and so the holomorphic tension is a good approximation to the real tension if the following condition is satisfied:

$$\frac{e^2 W''^2}{W'} \ll 1 \,, \tag{5.7}$$

where we don't write the modulus for convenience. See [20] for a numerical computation or the first order correction that we have estimate above.

Now we consider quantum corrections to SQED. Being this theory infrared free, the coupling constant at low energy goes like

$$\frac{1}{e^2} \sim \log \frac{\Lambda_{\mathrm{U}(1)}}{\mu} \,, \qquad \mu \ll \Lambda_{\mathrm{U}(1)} \,. \tag{5.8}$$

The condition (5.7) becomes:

$$\frac{W''^2}{\log\left(\Lambda_{\mathrm{U}(1)}/\sqrt{W'}\right) W'} \ll 1 \,, \tag{5.9}$$

where $\sqrt{W'}$ is the energy scale of the U(1) breaking.

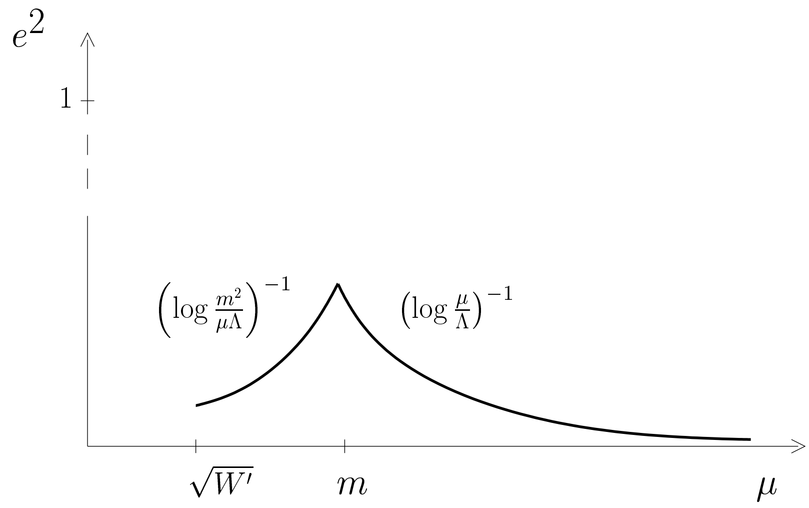

Figure 2: RG flow of e^2 at weak coupling ($m \gg \Lambda$).

5.2 U(N_c) theory with N_f flavors: semiclassical limit

Now we embed the U(1) theory in the asymptotically free (AF) theory U(N_c) with N_f flavors (see figure 2). The AF theory has a dinamical scale Λ, and the coupling constant at high energy goes like

$$\frac{1}{e^2} \sim \log \frac{\mu}{\Lambda}, \qquad \mu \gg \Lambda\,. \tag{5.10}$$

To find $\Lambda_{\mathrm{U}(1)}$ one has to match (5.8) with (5.10) when $\mu = m$. Note that this is reliable only if $m \gg \Lambda$ and so in the weak coupling regime. With this matching we obtain $\Lambda_{\mathrm{U}(1)} \sim m^2/\Lambda$ and the condition (5.9) becomes:

$$\frac{W''^2}{\log\left(m^2/\Lambda\sqrt{W'}\right)W'} \ll 1\,. \tag{5.11}$$

5.3 Type I or type II?

The BPS tension scales linearly with the winding number n

$$T_{\mathrm{BPS}} = 4\pi|n\mathcal{T}|\,. \tag{5.12}$$

What about the real tension? Considering (5.1), we can say exactly that the real tension is less than or equal to the BPS tension $T(n) \leq 4\pi|n\mathcal{T}|$. This is because the BPS vortex is not a solution of the equations of motions and so it is not the one that minimizes the tension (5.1). Note also that the first two corrections (5.4) and (5.5) are positive, which means that the last one (5.6) is negative and must dominate.

To decide whether the vortex is of type I or type II, one should verify how the first order correction scales with n. Looking at the equations for the profile functions (2.9), we see that R_v grows with n, but this information is not enough to establish whether the sum of the three corrections grows or decay, because they scale differently with R_v (compare for example (5.5) with (5.6)). To get this information we must know the specific superpotential (see for example [12] and [21]). If the correction grows with n then $T(2) < 2T(1)$ and the vortices are of type I, on the contrary they are of type II.

6. Summary and discussion

Here we summarize the main result of this paper. We have studied the tension of vortices that arise in color-flavor locking vacua of $\mathcal{N}=2$ SQCD broken to $\mathcal{N}=1$ by a superpotential. The tension can be written as

$$T = 4\pi|\mathcal{T}| + T_{\text{non BPS}}\,, \tag{6.1}$$

where $\mathcal{T}$ is the holomorphic tension and is related to the central charge of the theory. To compute $\mathcal{T}$ we must first solve the factorization equations:

$$y^2 = P_{N_c}(z)^2 - \Lambda^{2N_c-N_f}\prod_{I=1}^{N_f}(z-m_I) = F_{2n}(z)H_{N_c-n}(z)^2 \tag{6.2}$$

and

$$y_m{}^2 = W'(z)^2 + f(z) = g_k{}^2 F_{2n}(z)Q_{k-n}(z)^2\,. \tag{6.3}$$

These equations give, as a function of $W(z)$ and Λ, the points of the moduli space that survive after the perturbation and the polynomial $f(z)$. In particular we obtain $\widetilde{m}_I$, that is the double root of $H_{N_c-n}(z)^2$ in (6.2) that in the semiclassical limit becomes m_I. With these results we can compute the holomorphic tension of the I-vortex:

$$\mathcal{T}_I = \sqrt{W'^2+f}\,\Big|_{z=\widetilde{m}_I}\,. \tag{6.4}$$

What is the nature of the quantum corrections to the holomorphic tension? The corrections comes from $\widetilde{m}$ and f and both of them are computed using the factorization equations (6.2) and (6.3). When $\Lambda=0$ the computation gives $\widetilde{m}=m$ and $f=0$, thus the corrections are positive powers of Λ. Inserting in (6.4) we can expand in powers of Λ/m:

$$\mathcal{T} = W'(m) + \sum_{l=1}^{\infty}\mathcal{T}_l\left(\frac{\Lambda}{m}\right)^l\,. \tag{6.5}$$

The terms $\mathcal{T}_l$ can be interpreted, in the semiclassical limit, as an l-instantons correction to the holomorphic tension and the perturbative corrections are absent.

In the semiclassical region our vortices are created by the winding of the quarks Q, $\widetilde{Q}$ and so they carry magnetic flux. In the strong coupling region the vortices are created by the winding of dual quarks and so they carry also electric flux. As m is varied and reaches values below the dynamical scale Λ of the theory the change of monodromy around the quark singularity occurs, when it moves below the cuts produced by other singularities. Quarks may become magnetic monopoles [2]. The precise way this type of metamorphosis takes place has been studied explicitly only in the simplest cases [22]. For this reason we believe that, in the same way quarks are continuously changed in the dual quarks, the color-locked vortices are continuously connected if we move the parameter m. The same problem was encuontered in [23] dealing with non-abelian magnetic monopoles.

There is also a non-BPS contribution to the tension that we are not able to compute. At least in the semiclassical limit, we have found that the non-BPS contribution can be neglected when the condition (5.11) holds. It would be interesting to find a global defined (i.e. defined for every value of m) condition

$$F(g_j, m, \Lambda) \ll 1 \tag{6.6}$$

to say when $T_{\mathrm{non\,BPS}} \ll T_{\mathrm{BPS}}$. This condition should becomes (5.11) at weak coupling. Another interesting question is: can one find a global defined condition to distinguish whether the vortices are of type I or type II?

Our analysis has been performed in the strong coupling regime where there are two different scales of symmetry breaking

$$\mathrm{U}(N_c) \longrightarrow \mathrm{U}(1)^{N_c} \longrightarrow \mathrm{U}(1)^n \,. \tag{6.7}$$

The topological stability of the $N_c - n$ vortices is assured only at the lower energy scale by the homotopy group $\pi_1(\mathrm{U}(1)^{N_c-n}) = \mathbb{Z}^{N_c-n}$. If we relax the strong coupling condition a vortex can decay in a lighter one. The topology of the global U(1), the one that comes from the decomposition $\mathrm{U}(N_c) = \mathrm{U}(1) \times \mathrm{SU}(N_c)$, assures at least the stability of the lighter vortex.

Moreover, there is a another issue we want to discuss. The color-flavor locking points are not the only ones where there are massless charged particles that condense and create a vortex. Consider, for example, pure $\mathrm{U}(N_c)$ theory [1, 24], in which there are points where some dyons become massless and condense. What about the tension in this case? The analysis at strong coupling given in section 4 is still valid. In this case one cannot bring these kind of vacua at weak coupling because the dyons are always massive in this regime.

Finally we consider the simplest example $n = k = 0$, where the superpotential is linear

$$W(z) = g_0 z \,. \tag{6.8}$$

As $W'' = 0$, there is no lost of information in rotating away the superpotential so that it becomes an FI term for the global U(1): $-\int d^2\theta d^2\bar{\theta} 2r \mathrm{Tr}_{N_c} V$. This is the same model considered in [25] and [26].[4] Classically the vacua that survives are the ones completely locked, so we need at least N_c flavors. The holomorphic tension is

$$\mathcal{T} = g_0 \tag{6.9}$$

and the non-BPS contribution is absent. Our result is particularly powerful in this case, in fact the polynomial $f(z)$, being of degree $z-1$, is zero. So there are no quantum corrections to (6.9) and also the non-BPS contribution is absent for every value of m. Thus the vortices are exactly BPS and the tension is exactly $T = 4\pi|g_0|$.

[4] Similar vortices are found also in supersymmetric theories in six spacetime dimensions with fundamental hypermultiplets [27].

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A. Details on the central charge

The holomorphic tension is strictly related to the central charge of the theory, as in the case of the dyon mass in $\mathcal{N}=2$. Here we review the tools to study the central charge in the presence of a vortex [6]. In the presence of a vortex the $\mathcal{N}=1$ superalgebra is modified by of a term that breaks the Lorentz invariance

$$\begin{aligned} \{Q_\alpha, \overline{Q}_{\dot\alpha}\} &= 2P_{\alpha\dot\alpha} + 2Z_{\alpha\dot\alpha}\,, \\ \{Q_\alpha, Q_\alpha\} &= 0\,, \qquad \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\alpha}\} = 0\,. \end{aligned} \tag{A.1}$$

The central charge Z_μ is proportional to the vortex orientation[5] n_μ multiplied by its length L

$$Z_\mu = TLn_\mu\,, \tag{A.2}$$

where T is a constant of proportionality. Now we go in the rest frame of this object (the piece of vortex of length L), where $P^\mu = (M,0,0,0)$ and, orienting it in the $\hat z$ direction, $n^\mu = (0,0,0,1)$. The algebra becomes

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2\begin{pmatrix} M+TL & \\ & M-TL \end{pmatrix}_{\alpha\dot\alpha}. \tag{A.3}$$

When the mass saturates the bound

$$M = TL\,, \tag{A.4}$$

half of the supersymmetries are unbroken, and so we see that T is the BPS tension.

The central charge depends on the theory and to calculate it one uses the supercurrent $\overline{S}_{\mu\dot\alpha}$ that corresponds to the generator

$$\overline{Q}_{\dot\alpha} = \int d^3x\, \overline{S}_{0\dot\alpha}\,. \tag{A.5}$$

We calculate the supersymmetric variation of this current

$$\delta_\alpha \overline{S}_{\nu\dot\alpha} = 2\sigma^\rho{}_{\alpha\dot\alpha} T_{\nu\rho} + \partial^\rho R_{\nu\rho\,\alpha\dot\alpha}\,, \tag{A.6}$$

where $T_{\nu\rho}$ is the energy-momentum tensor and $\partial^\rho R_{\nu\rho\,\alpha\dot\alpha}$ is a boundary term that satisfy

$$R_{\nu\rho\,\alpha\dot\alpha} = -R_{\rho\nu\,\alpha\dot\alpha}\,. \tag{A.7}$$

[5] n_μ is the vortex spatial orientation in the rest frame.

Integrating (A.6) we obtain the superalgebra

$$\int d^3x\, \delta_\alpha \overline{S}_{0\dot\alpha} = \left\{ Q_\alpha, \int d^3x\, \overline{S}_{0\dot\alpha} \right\} = 2P_{\alpha\dot\alpha} + 2Z_{\alpha\dot\alpha}\,. \tag{A.8}$$

Thus, the central charge is given by the integral of the boundary term in (A.6)

$$2Z_{\alpha\dot\alpha} = \int d^3x\, \partial^i R_{0i\,\alpha\dot\alpha}\,. \tag{A.9}$$

To compute R, it is convenient to go in bispinorial notation.[6] The variation (A.6) in bispinorial notation is

$$\delta_\alpha \overline{S}_{\beta\dot\beta\dot\alpha} = 2T_{\beta\dot\beta\,\alpha\dot\alpha} + \partial^\rho R_{\beta\dot\beta\,\rho\,\alpha\dot\alpha}\,. \tag{A.10}$$

The two terms can be distinguished by symmetry: the energy momentum tensor contains the terms both symmetric in $\alpha\,\beta$ and $\dot\alpha\,\dot\beta$ or both antisymmetric, while R contains the terms with mixed symmetry. Now we consider an $\mathcal{N} = 1$ SQED with gauge multiplet A_μ, λ and some charged chiral multiplets q_i, ψ_1 of charge t_i. The computation of R in this model gives

$$R_{\beta\dot\beta\ \ \alpha\dot\alpha}^{\ \ \ \rho} \propto \sum_i \left(\epsilon_{\dot\alpha\dot\beta}\sigma^{\rho\nu}{}_{\alpha\beta} + \epsilon_{\alpha\beta}\bar\sigma^{\rho\nu}{}_{\dot\alpha\dot\beta}\right) A_\nu q_i^\dagger t_i q_i + \text{ferm}\,. \tag{A.11}$$

We don't need to consider the fermion terms because in the vortex solution only the bosonic fields are excited. Coming back to vector notation, we get

$$R_{\nu\rho\mu} = \sum_i \epsilon_{\nu\rho\mu\tau} A^\tau q_i^\dagger t_i q_i \tag{A.12}$$

and (A.9) becomes:

$$Z_0 = 0\,, \qquad Z_j \propto \int d^3x \sum_i \partial^l (\epsilon_{ljk} A_k q_i^\dagger t_i q_i)\,. \tag{A.13}$$

Remembering (A.2) one obtains the tension of the vortex

$$T_{\text{BPS}} = \oint d\vec{x} \cdot \vec{A} \sum_i q_i^\dagger t_i q_i\,. \tag{A.14}$$

The term that appears in the BPS tension is the same that appears in the D term of the potential

$$V_D = \frac{e^2}{2}\left(\sum_i q_i^\dagger t_i q_i - 2r\right)^2, \tag{A.15}$$

where we have considered the option of a Fayet-Iliopulos term in the lagrangian:

$$-\int d^2\theta d^2\bar\theta\, 2rV\,. \tag{A.16}$$

The FI term is crucial because without it the central charge would be zero, in fact from (A.14) and (A.15) the BPS tension is proportional to the FI term:

$$T_{\text{BPS}} = 2r \oint d\vec{x} \cdot \vec{A}\,. \tag{A.17}$$

[6]If v_μ is a vector, the passage is given by these formulas: $v_{\alpha\dot\alpha} = \sigma^\mu{}_{\alpha\dot\alpha} v_\mu$ and $v^\mu = -\frac{1}{2}\bar\sigma^{\mu\alpha\dot\alpha} v_{\alpha\dot\alpha}$.

B. Note on conventions

In the conventions that we use in the paper (3.1)

$$W^{\alpha}W_{\alpha}|_{\theta\theta} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2i\bar{\lambda}\bar{\sigma}^{\mu}D_{\mu}\lambda + D^2 + \frac{i}{4}\epsilon_{\mu\nu\rho\tau}F_{\mu\nu}F_{\rho\tau}\,. \tag{B.1}$$

To have the same normalization of [10] we should use the following definitions:

$$T(z) = \mathrm{Tr}\frac{1}{z-\Phi}\,, \qquad R(z) = -\frac{1}{16\pi^2}\mathrm{Tr}\,\frac{W^{\alpha}W_{\alpha}}{z-\Phi}\,, \tag{B.2}$$

$$M_I(z) = \widetilde{Q}_I\frac{1}{z-\Phi}Q_I\,. \tag{B.3}$$

Taking into account that our superpotential is $\sqrt{2}(\widetilde{Q}\Phi Q - m\widetilde{Q}Q + W(\Phi))$, the generalized Konishi anomalies of [10] become:

$$\begin{aligned}[\sqrt{2}W'(z)R(z)]_{-} &= R(z)^2\,,\\ [M_I(z)\sqrt{2}(z-m_I)]_{-} &= R(z)\,.\end{aligned} \tag{B.4}$$

In the article we use, instead of this, the convection (3.5) for $R(z)$, such that (B.4) are simplified and become (3.7).

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